

Exercise 56

Evaluate $\lim_{t \rightarrow 0} \frac{t^3}{\tan^3(2t)}$.

Solution

Recall that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1.$$

Now evaluate the given limit.

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{t^3}{\tan^3 2t} &= \lim_{t \rightarrow 0} \frac{t^3}{\frac{\sin^3 2t}{\cos^3 2t}} \\ &= \lim_{t \rightarrow 0} \left(\frac{t^3}{\sin^3 2t} \right) \cos^3 2t \end{aligned}$$

Make the substitution $u = 2t$, or $\frac{u}{2} = t$. Then as $t \rightarrow 0$, $u \rightarrow 0$ as well.

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{t^3}{\tan^3 2t} &= \lim_{u \rightarrow 0} \left[\frac{\left(\frac{u}{2}\right)^3}{\sin^3 u} \right] \cos^3 u \\ &= \lim_{u \rightarrow 0} \left(\frac{\frac{u^3}{8}}{\sin^3 u} \right) \cos^3 u \\ &= \lim_{u \rightarrow 0} \frac{1}{8} \left(\frac{u^3}{\sin^3 u} \right) \cos^3 u \\ &= \frac{1}{8} \lim_{u \rightarrow 0} \left(\frac{u}{\sin u} \right)^3 (\cos u)^3 \\ &= \frac{1}{8} \left(\lim_{u \rightarrow 0} \frac{u}{\sin u} \right)^3 \left(\lim_{u \rightarrow 0} \cos u \right)^3 \\ &= \frac{1}{8} (1)^3 (\cos 0)^3 \\ &= \frac{1}{8} \end{aligned}$$